Solow: It is all about physical capital accumulation

Felix Wellschmied

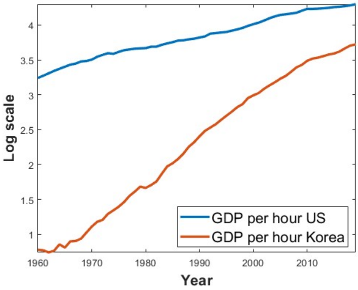
Universidad Carlos III de Madrid

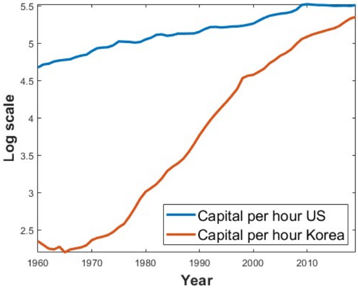
Economic growth: Theory and Empirical Methods

Modern economic growth

- You have seen that over the last 150 years, economic growth in advanced economies can be described by exponential growth.
- You have seen that there exist huge cross-country differences in income per person across countries.
- Income differences are not stable. In fact, there are some "growth miracles".
- Growth miracles are associated with rapid capital accumulation.

Korea, a growth miracle





The importance of growth miracles

"I do not see how one can look at figures like these without seeing them as representing *possibilities*. Is there some action the government of India could take that would lead the Indian economy to grow like Indonesia's or Egypt's? [...] The consequences for human welfare involved in questions like these are simply staggering: Once one starts to think about them, it is hard to think about anything else." Lucas Jr (1988)

Modern economic growth

- To understand these phenomena, we need a theory.
- Our theory will be guided by certain data facts.
- These data facts should be "universally" true, i.e., relatively stable over time.

Facts about modern economic growth: Kaldor facts

Kaldor (1961) summarized six facts about income. We will consider the first five:

1. Output per worker grows at a constant rate over time.
2. Capital per worker grows at a constant rate over time.
3. The capital-to-output ratio is constant over time.
4. Capital has a constant rate of return over time.
5. The share of income going to capital is constant over time.

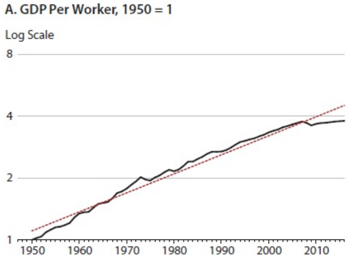
Facts about modern economic growth: Kaldor facts II

Recently, Herrendorf et al. (2019) consider these facts anew including recent data:

1. Broadly speaking, the Kaldor facts still hold.
2. However, some data moments show some time variation.
3. In this course, we will, nevertheless, use the Kaldor facts as the benchmark.

Constant growth in output per worker

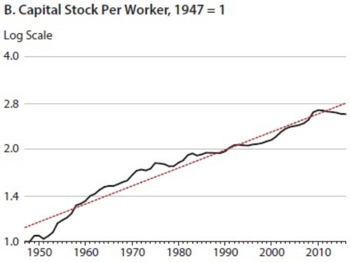
U.S.



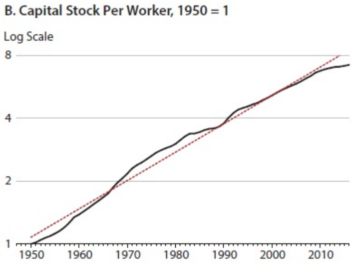
UK

- A constant output per worker growth is a good approximation.
- However, we observe a slow-down after 1970.

Constant growth in capital per worker



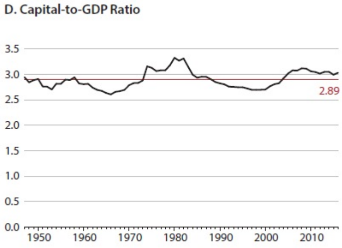
U.S.



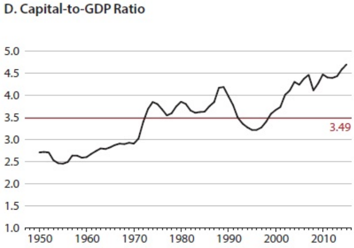
UK

- A constant capital per worker growth is a good approximation.
- However, we observe a slow-down after 1970.

A constant capital-to-output ratio



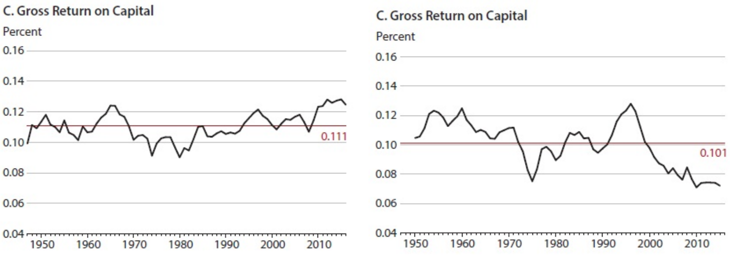
U.S.



UK

- In the U.S., capital and output grow approximately at the same rate.
- In the UK, capital grows faster than output.

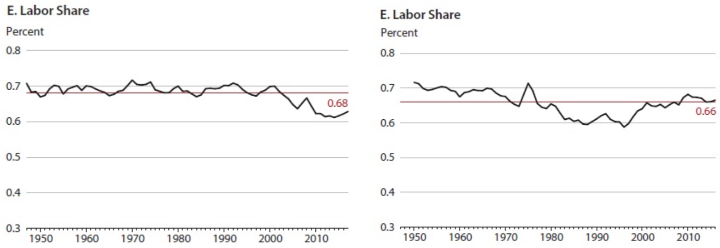
Constant return to capital



U.S. UK

- A constant return to capital is a good approximation.
- Since the 2000s, we observe some time variation that is different across countries.

A constant capital share in income



U.S. UK

- Instead of the capital share, they consider the easier to measure labor share.
- The labor share started to fall in the U.S. during the 2000s. It was falling in the UK between the 50s and 2000s.

Understanding modern economic growth

A first attempt to understand modern economic growth

- Solow (1956) presents a framework on how to understand the phenomenon of modern economic growth (Kaldor factors). For that work, he won the Nobel price.
- It is a closed economy model where production takes place by labor, capital, and technology.
- Importantly, it takes technological growth as exogenous and puts physical capital accumulation center stage.

How production takes place

The real world

- In a modern economy, production takes place mostly at firms which are often multinational corporations.
- These firms produce thousands of different goods and services relying on thousands on imports and creating thousands of exports.
- For production, they employ
  - labor of different types (education, age, sex...)
  - equipment, structures, roads, land, raw materials...
- A lot of production is also done by the government.
- Most exchange of goods and services as well as factor inputs is conducted in thousands of markets.
- People make decisions about consumption today versus the future.

Abstractions

The world is quite complicated (more so than medieval England) and we will have to make simplifications to make progress in understanding it:

- We assume a closed economy.
- We abstract from the government and treat it just as the private sector.
- There is only one output good.
- Production takes place at the level of firms that rent the factors of production from households.

Abstractions: markets

To understand how firms and households interact, we look at data from national accounts:

- The profit share of national income is relatively small, around 5%.
- This suggests that product markets and input markets are close to perfectly competitive.

This implies that the factors of production earn their marginal products.

Abstractions: factors of production

To focus on the right factor inputs, we look again at national accounts:

- As we have seen before, labor compensation is around 2/3 of national income, i.e., it is important, and we include labor into the model.
- The second large recipient of national income is capital which we include into the model. We treat land as capital. One may object that land is finite. However, fertilizers and tall buildings suggest otherwise.
- Natural resources are different from capital because they are finite. Even for the U.S., a major oil producer, income from natural resources is relatively small. Hence, we will ignore them. One can also interpret them as physical capital recognizing that, so far, their supply did not run out.

Abstractions: aggregation of factors of production

To make the model tractable, we assume that we can aggregate labor and capital:

- We aggregate labor across occupations and worker skills.
- We aggregate capital across all types of capital goods.

Abstractions: technology

We also assume a single measure on how well we use capital and labor to produce output, i.e., technology.

- This may relate to firm organization, e.g., management style.
- This may relate to logistics, e.g., just-in-time delivery.
- This may relate to new products that are better than the old product but not more expensive to produce, e.g., a faster computer algorithm.
- In fact, many of the products we consume today did not exist 50 years ago.
- Hence, we will think of improvements in A as new *ideas*, or better recipes.

Abstractions: The aggregate production function

Our assumptions imply that firms operate an aggregate production function that combines a single labor input, L, a single capital input, K, and some technology level, A, into a single output good: $Y = F(K, L, A)$. Our model should be consistent with the Kaldor facts. We now use the fact of constant income shares to figure out how F should look like:

$$\frac{r(t)K(t)}{Y(t)} = \alpha, \tag{1}$$

$$\frac{w(t)L(t)}{Y(t)} = 1 - \alpha. \tag{2}$$

Given the assumption of competitive markets, we have:

$$\frac{\frac{\partial Y(t)}{\partial K(t)}K(t)}{Y(t)} = \alpha, \tag{3}$$

$$\frac{\frac{\partial Y(t)}{\partial L(t)}L(t)}{Y(t)} = 1 - \alpha. \tag{4}$$

Abstractions: The aggregate production function II

This holds for, among others, the Cobb-Douglas production function:

$$Y(t) = K(t)^{\alpha} (A(t)L(t))^{1-\alpha} \tag{5}$$

Note, the place of $A(t)$ in the production function is not particularly important:

$$Y(t) = K(t)^{\alpha} (A(t)L(t))^{1-\alpha} = A(t)^{1-\alpha} K(t)^{\alpha} L(t)^{1-\alpha} = E(t) K(t)^{\alpha} L(t)^{1-\alpha}, \tag{6}$$

with $E(t) = A(t)^{1-\alpha}$. The way I have written it above will make the math easier.

Abstractions: The aggregate production function III

One key property of the aggregate production function are diminishing marginal returns to the factor inputs:

$$\frac{\partial^2 Y(t)}{\partial^2 L(t)} = -\alpha(1-\alpha)K(t)^{\alpha}A(t)^{1-\alpha}L(t)^{-\alpha-1} < 0 \quad (7)$$

$$\frac{\partial^2 Y(t)}{\partial^2 K(t)} = (\alpha-1)\alpha K(t)^{\alpha-2}\left(A(t)L(t)\right)^{1-\alpha} < 0 \quad (8)$$

$$\quad (9)$$

This is a natural outcome when aggregating factors of production: More units of the same factor input lose marginal productivity as their numbers increase. Aggregating factors of production was heavily criticized by Robinson (1953) and was part of the heated Cambridge-Cambridge debate (for example, Solow (1955)) during the 50s.

Aggregating factors of production and level of technology

It turns out, that aggregation matters a lot when thinking about technological progress:

- Managers trained at business schools make low-skilled labor more, not less, productive.
- Introducing drones make tractors more, not less, productive.

In the Solow model, such a diversification of factors of production is part of technological progress.

Abstractions: Household decisions

- One of the most important questions in modern macroeconomics is how households trade-off consumption today against tomorrow.
- The Solow model abstracts from this and assumes that households save a constant fraction of their income each period.

What do we (not) explain?

We will take households' savings rates, investment into education, and population growth rates as given. As you have seen, to explain differences in those, we need to understand

- cultural differences between societies.
- geographic differences between societies.
- institutional differences between societies.

Solving the model

Production vs. income

To know households' decisions, we need to know their income. The prices for their factors of production are:

$$w(t) = \frac{\partial Y(t)}{\partial L(t)} = (1-\alpha)K(t)^{\alpha}A(t)^{1-\alpha}L(t)^{-\alpha} \tag{10}$$

$$r(t) = \frac{\partial Y(t)}{\partial K(t)} = \alpha K(t)^{\alpha-1}\left(A(t)L(t)\right)^{1-\alpha} \tag{11}$$

$$\tag{12}$$

Hence, as households own the factors of production, total household income is

$$r(t)K(t) + w(t)L(t) = Y(t). \tag{13}$$

This will be convenient, as we do not need to distinguish between production and household income. That is, aggregate savings are simply $S(t) = sY(t)$.

Capital accumulation

The Solow model assumes that every period a fraction δ of the capital stock depreciates. Working against this, households invest $I(t) = S(t)$:

$$\dot{K}(t) = S(t) - \delta K(t) \tag{14}$$

$$\dot{K}(t) = sY(t) - \delta K(t) \tag{15}$$

$$\dot{K}(t) = sK(t)^{\alpha} \left(A(t)L(t)\right)^{1-\alpha} - \delta K(t). \tag{16}$$

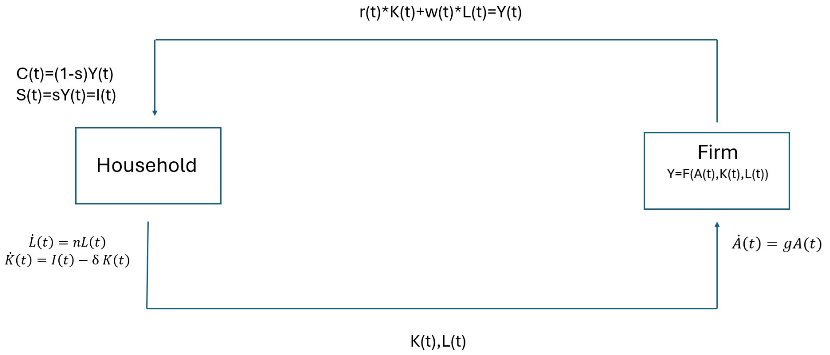
Dynamics of population and technology

The Solow model assumes that the population and technology grow at exogenous rates. To be consistent with the Kaldor facts on labor productivity, it assumes they grow exponentially:

$$L(t) = L(0)\exp(nt) \Rightarrow \frac{\dot{L}(t)}{L(t)} = n \tag{17}$$

$$A(t) = A(0)\exp(gt) \Rightarrow \frac{\dot{A}(t)}{A(t)} = g. \tag{18}$$

Summary of the economy



Steady state

As before, we will start our analysis with the steady state of the model. For this, we need to find a variable that has a steady state. It turns out, in the Solow model this is the capital-to-output ratio which is consistent with a constant ratio over time in the data. To find the steady state, use the production function to write:

$$z(t) = \frac{K(t)}{Y(t)} = \frac{K(t)}{K(t)^{\alpha} \left(A(t)L(t)\right)^{1-\alpha}} \tag{19}$$

$$= \left(\frac{K(t)}{A(t)L(t)}\right)^{1-\alpha}. \tag{20}$$

Growth rate of the capital-to-output ratio

As we are interested in its dynamics, we write this in terms of its growth rate using that the derivative of a variable in logs with respect to time is the growth rate of that variable:

$$\ln z(t) = (1-\alpha)\ln K(t) - (1-\alpha)(\ln L(t) + \ln A(t)) \tag{21}$$

$$\Rightarrow \frac{\dot{z}(t)}{z(t)} = (1-\alpha)\frac{\dot{K}(t)}{K(t)} - (1-\alpha)\left(\frac{\dot{L}(t)}{L(t)} + \frac{\dot{A}(t)}{A(t)}\right) \tag{22}$$

$$\frac{\dot{z}(t)}{z(t)} = (1-\alpha)\frac{\dot{K}(t)}{K(t)} - (1-\alpha)\left(n+g\right). \tag{23}$$

Growth rate of the capital-to-output ratio II

Conjecture that in steady state with $\dot{z}(t) = 0$ exists:

$$0 = (1-\alpha)\left(\frac{\dot{K}(t)}{K(t)}\right)^{*} - (1-\alpha)(n+g) \tag{24}$$

$$\left(\frac{\dot{K}(t)}{K(t)}\right)^{*} = n + g. \tag{25}$$

If a steady state exists where the capital-to-output ratio is constant, capital needs to grow at rate $n + g$.

Rewriting the capital accumulation equation

We need to find an expression for the growth rate of the capital stock

$$\dot{K}(t) = sK(t)^{\alpha}\left(A(t)L(t)\right)^{1-\alpha} - \delta K(t) \tag{26}$$

$$\frac{\dot{K}(t)}{K(t)} = \frac{s}{z(t)} - \delta \tag{27}$$

Combining the equations and evaluating at steady state:

$$n + g = \frac{s}{z^*} - \delta \tag{28}$$

$$z^* = \left(\frac{K(t)}{Y(t)}\right)^* = \frac{s}{n+g+\delta}. \tag{29}$$

Solving for the steady state

$$z^* = \left(\frac{K(t)}{Y(t)}\right)^* = \frac{s}{n+g+\delta}. \tag{30}$$

Note, our variable, z^*, depends only on time-invariant parameters. Hence, if a steady state exists, we have found it. Note, we do not know yet whether $K(t)$ and $Y(t)$ are constant in steady state (they are not), we only know that their ratio is constant.

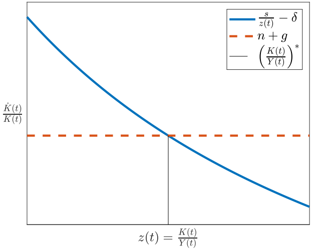
Capital-to-output ratio in steady state

$$\left(\frac{K(t)}{Y(t)}\right)^* = \frac{s}{n+g+\delta}. \tag{31}$$

The capital-to-output ratio in steady state is

- increasing in the savings rate.
- decreasing in the capital depreciation rate.
- decreasing in the population growth rate.
- decreasing in the technological growth rate.

The steady state graphically



Two steady state conditions:

$$\frac{\dot{K}(t)}{K(t)} = \frac{s}{z(t)} - \delta \tag{32}$$

$$\left(\frac{\dot{K}(t)}{K(t)}\right)^* = n + g \tag{33}$$

Note, there exist one steady state with $\frac{K(t)}{Y(t)} > 0$.

Output per capita in steady state

Start with the production function:

$$Y(t) = K(t)^{\alpha}\left(A(t)L(t)\right)^{1-\alpha} \tag{34}$$

$$\frac{Y(t)}{Y(t)^{\alpha}} = \left(\frac{K(t)}{Y(t)}\right)^{\alpha}\left(A(t)L(t)\right)^{1-\alpha} \tag{35}$$

$$Y(t) = \left(\frac{K(t)}{Y(t)}\right)^{\frac{\alpha}{1-\alpha}} A(t)L(t) \tag{36}$$

$$Y(t)^{*} = \left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} A(t)L(t) \tag{37}$$

$$y(t)^{*} = \left(\frac{Y(t)}{L(t)}\right)^{*} = \left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} A(t) \tag{38}$$

Output per capita in steady state II

$$\left(\frac{Y(t)}{L(t)}\right)^* = \left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} A(t) \tag{39}$$

Different from the Malthus model, the Solow model can explain long-run differences in income per capita:

- A higher technology level increases output per capita. However, this is just an exogenous process, i.e., the model does not help us to understand differences across countries.
- The endogenous part that the model helps us to understand are differences in the capital-to-output ratios across countries.

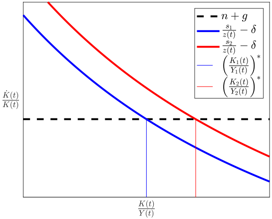
Output per capita in steady state III

$$\left(\frac{Y(t)}{L(t)}\right)^* = \left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} A(t) \quad (40)$$

A country is rich because of a high capital-to-output ratio because is has a

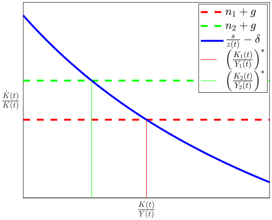
- high savings rate.
- a low population growth rate or capital depreciation rate.

Comparative statics: An increase in the savings rate



Consider an increase in the savings rate. In steady state, graphically, for any level of $\frac{K(t)}{Y(t)}$, $\frac{\dot{K}(t)}{K(t)} = \frac{s}{z(t)} - \delta$ increases. The new steady state is associated with a higher $\left(\frac{K(t)}{Y(t)}\right)^*$ and, hence, a higher $y(t)^* = \left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} A(t)$.

Comparative statics: An increase in the population growth rate



Consider an increase in the population growth rate. In steady state, graphically, for any level of $\frac{K(t)}{Y(t)}$, $n+g$ increases. The new steady state is associated with a lower $\left(\frac{K(t)}{Y(t)}\right)^*$ and, hence, a lower $y(t)^* = \left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} A(t)$.

Consumption per capita in steady state

Once we know output (per capita), knowing consumption is simple as it is just a constant fraction of output:

$$C(t) = (1 - s)Y(t) \tag{41}$$

Hence, consumption per worker in steady state is

$$\left(\frac{C(t)}{L(t)}\right)^* = (1 - s)\left(\frac{s}{n + g + \delta}\right)^{\frac{\alpha}{1-\alpha}} A(t). \tag{42}$$

Capital per capita in steady state

Start with the steady state capital-to-output ratio

$$\left(\frac{K(t)}{Y(t)}\right)^* = \frac{s}{n+g+\delta} \tag{43}$$

and plug in the production function:

$$\left(\frac{K(t)}{K(t)^\alpha \left(A(t)L(t)\right)^{1-\alpha}}\right)^* = \frac{s}{n+g+\delta} \tag{44}$$

$$\left(\frac{K(t)^{1-\alpha}}{L(t)^{1-\alpha}}\right)^* = \frac{s}{n+g+\delta}A(t)^{1-\alpha} \tag{45}$$

Hence, in steady state:

$$\left(\frac{K(t)}{L(t)}\right)^* = \left(\frac{s}{n+g+\delta}\right)^{\frac{1}{1-\alpha}} A(t). \tag{46}$$

The rental price of capital in steady state

The rental price of capital is given by

$$r(t) = \frac{\partial Y(t)}{\partial K(t)} = \alpha K(t)^{\alpha-1} \left(A(t)L(t)\right)^{1-\alpha} = \frac{\alpha}{\left(\frac{K(t)}{Y(t)}\right)} \tag{47}$$

$$r^* = \frac{\alpha}{\left(\frac{K(t)}{Y(t)}\right)^*} = \alpha \frac{n+g+\delta}{s}, \tag{48}$$

which is a constant in the long run. Hence, the Solow model is consistent with the Kaldor fact on constant returns to capital.

Wages in steady state

The wage is given by

$$w(t) = \frac{\partial Y(t)}{\partial L(t)} = (1-\alpha)A(t)\left(\frac{K(t)}{A(t)L(t)}\right)^{\alpha} \tag{49}$$

$$w(t) = (1-\alpha)A(t)^{1-\alpha}\left(\frac{K(t)}{L(t)}\right)^{\alpha} \tag{50}$$

$$w(t) = (1-\alpha)A(t)\left(\frac{K(t)}{Y(t)}\right)^{\frac{\alpha}{1-\alpha}} \tag{51}$$

$$w(t)^{*} = (1-\alpha)A(t)\left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} \tag{52}$$

which is growing at the rate g in steady state. Kaldor did not study wages but this fact is also born out by the data.

Factor shares in steady state

The shares of income going to capital and labor, respectively, are:

$$\frac{r(t)K(t)}{Y(t)} = \alpha K(t)^{\alpha} (A(t)L(t))^{1-\alpha} = \alpha Y(t) \tag{53}$$

$$\frac{w(t)L(t)}{Y(t)} = (1-\alpha)K(t)^{\alpha} (A(t)L(t))^{1-\alpha} = (1-\alpha)Y(t). \tag{54}$$

That is, capital obtains α of output and labor obtains $1-\alpha$ of output. Hence, the Solow model is also consistent with the Kaldor facts about constant factor shares.

Growth of capital in steady state

In steady state, by definition, the capital-to-output ratio is constant. This does not mean, however, that capital or capital per worker, are constant. In fact, we already know that:

$$\frac{\dot{z}(t)}{z(t)} = (1-\alpha)\frac{\dot{K}(t)}{K(t)} - (1-\alpha)(n+g) \tag{55}$$

$$\left(\frac{\dot{K}(t)}{K(t)}\right)^* = n+g \tag{56}$$

$$\left(\frac{\dot{k}(t)}{k(t)}\right)^* = g. \tag{57}$$

That is, in steady state, capital per capita grows at the rate of technological progress. A constant growth rate of capital per capita is one of the Kaldor facts.

Intuition: Why is capital growing in steady state

Given our production function, the capital to output ratio is proportional to the marginal product of capital:

$$\frac{K(t)}{Y(t)} = \frac{K(t)}{K(t)^{\alpha}\left(A(t)L(t)\right)^{1-\alpha}} = \frac{\alpha}{MPK(t)}. \tag{58}$$

As $MPK(t)$ measures the productivity of capital, this relationship allows us to better understand the economic intuition of the model.

Intuition: Why is capital growing in steady state II

From the capital accumulation equation:

$$\frac{\dot{K}(t)}{K(t)} = \frac{sY(t)}{K(t)} - \delta = \frac{s}{\frac{K(t)}{Y(t)}} - \delta \tag{59}$$

$$= \frac{s}{\alpha} MPK(t) - \delta \tag{60}$$

As technology and labor are growing, ceteris paribus, the marginal product of capital is growing. A higher marginal product pushes investment per unit of capital beyond the depreciation rate of that capital (δ), leading to additional capital growth. The additional capital decreases again its marginal product, leading to a constant marginal product in steady state:

$$MPK^* = \frac{\alpha}{\left(\frac{K(t)}{Y(t)}\right)^*} = \alpha \frac{n + g + \delta}{s}. \tag{61}$$

Growth of output in steady state

The growth rate of output is

$$\frac{Y(t)}{Y(t)} = \frac{\alpha}{1-\alpha}\frac{\dot{z}(t)}{z(t)} + \frac{\dot{A}(t)}{A(t)} + \frac{\dot{L}(t)}{L(t)} \tag{62}$$

$$\left(\frac{\dot{Y}(t)}{Y(t)}\right)^{*} = n + g \tag{63}$$

$$\left(\frac{\dot{y}(t)}{y(t)}\right)^{*} = g. \tag{64}$$

Hence, output per capita in steady state also grows at the rate of technological progress, another Kaldor fact.

Intuition: Why is output growing in steady state

Writing the production function in growth rates yields

$$Y(t) = K(t)^{\alpha} \left(L(t)A(t)\right)^{1-\alpha} \tag{65}$$

$$\frac{\dot{Y}(t)}{Y(t)} = \alpha \frac{\dot{K}(t)}{K(t)} + (1-\alpha)\left(\frac{\dot{L}(t)}{L(t)} + \frac{\dot{A}(t)}{A(t)}\right). \tag{66}$$

The equation may suggest that output should grow at rate $(1-\alpha)(n+g) < n+g$. As we have diminishing marginal returns, an increase in $L(t)$ or $A(t)$ increases output by less than one unit. However, recall that capital is also growing endogenously at $n+g$. Together with constant returns to scale, we obtain a growth rate of output of $n+g$!

Growth of consumption in steady state

Finally, for consumption,

$$C(t) = (1 - s)Y(t) \tag{67}$$

$$\left(\frac{\dot{C}(t)}{C(t)}\right)^* = n + g \tag{68}$$

$$\left(\frac{\dot{c}(t)}{c(t)}\right)^* = g. \tag{69}$$

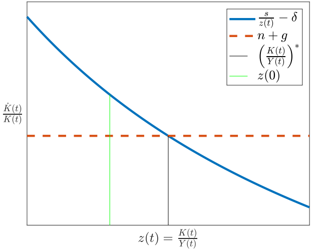
Hence, consumption per capita in steady state also grows at the rate of technological progress. A steady state in which all endogenous variables grow at the same rate is referred to as a *balanced growth path*.

What happens outside the steady state?

Hence, we have a model that is consistent with the Kaldor facts once the economy is in steady state. However, we would also like to understand how the economy behaves outside steady state:

- We still need to show that the steady state exists and, if it exists, whether the economy converges to its steady state.
- It allows us to study how the economy moves from one steady state to another if parameters change.
- In fact, we will see that outside steady state dynamics allow us to understand growth miracles.

Do we convergence to steady state?

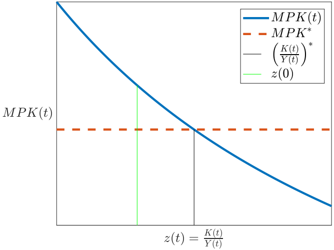


When $z(t) < z^*$, $\frac{\dot{K}(t)}{K(t)} = \frac{s}{z(t)} - \delta > n + g$ and, hence, the capital-to-output ratio is growing:

$$\frac{\dot{z}(t)}{z(t)} = (1-\alpha)\frac{\dot{K}(t)}{K(t)} - (1-\alpha)(n+g). \tag{70}$$

Hence, we converge to steady state from any starting point $z(0) > 0$.

Intuition for convergence to steady state



When $z(t) < z^*$, the marginal product of capital is higher than in the steady state. At high levels of the capital-to-output ratio, the reverse holds.

Intuition for convergence to steady state II

At low levels of the capital-to-output ratio, the marginal product of capital is high and, hence, the net return per unit of investment per capital is higher than the growth rate of output in steady state $(n+g)$, leading to an increase in the capital to output ratio:

$$\frac{s}{\alpha} MPK(t) - \delta > n + g \text{ if } z(t) < z^*. \tag{71}$$

At high levels of the capital-to-output ratio, the marginal product of capital is low and, hence, the net return per unit of investment per capital is lower than the growth rate of output in steady state $(n+g)$, leading to a decrease in the capital to output ratio:

$$\frac{s}{\alpha} MPK(t) - \delta < n + g \text{ if } z(t) > z^*. \tag{72}$$

Convergence to steady state

Using the dynamics of the capital-to-output ratio, and not imposing that the ratio is constant, we can solve for the convergence path explicitly:

$$\frac{\dot{z}(t)}{z(t)} = (1-\alpha)\frac{s}{z(t)} - (1-\alpha)(n+g+\delta) \tag{73}$$

$$\dot{z}(t) = (1-\alpha)s - (1-\alpha)(n+g+\delta)\,z(t) \tag{74}$$

Define $\beta = (1-\alpha)(n+g+\delta)$ and define

$$u(t) = s(1-\alpha) - \beta z(t) = \dot{z}(t) \tag{75}$$

with solution

$$u(t) = u(0)\exp(-\beta t) \tag{76}$$

Convergence to steady state II

Hence, we have

$$s(1-\alpha) - \beta z(t) = [s(1-\alpha) - \beta z(0)] \exp(-\beta t) \tag{77}$$

$$\underbrace{\frac{K(t)}{Y(t)}}_{\frac{\alpha}{MPK(t)}} - \underbrace{\frac{s}{n+g+\delta}}_{\left(\frac{K(t)}{Y(t)}\right)^*} = \left[\frac{K(0)}{Y(0)} - \frac{s}{n+g+\delta}\right] \exp(-\beta t). \tag{78}$$

- $\frac{K(t)}{Y(t)} - \frac{s}{n+g+\delta}$ converges to zero at rate $\beta = (1-\alpha)(n+g+\delta)$.
- In words: The absolute gap between the capital-to-output ratio and its steady state vanishes at rate β.

Shape of the convergence

To better understand the shape of the convergence, consider again the differential equation for $z(t) = \frac{K(t)}{Y(t)}$:

$$\dot{z}(t) = s(1-\alpha) - \beta z(t) \tag{79}$$

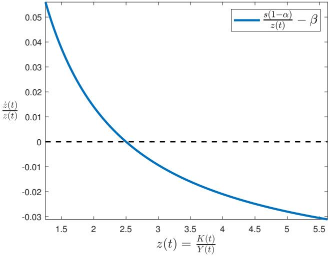
$$\frac{\dot{z}(t)}{z(t)} = \frac{s(1-\alpha)}{z(t)} - \beta \tag{80}$$

Note, the growth rate is 0 if $z(t) = \frac{s}{n+g+\delta} = z^*$. It is a decreasing, convex function in $z(t)$, and

$$z(t) \mapsto 0 \quad \frac{\dot{z}(t)}{z(t)} \mapsto \infty$$

$$z(t) \mapsto \infty \quad \frac{\dot{z}(t)}{z(t)} \mapsto \beta.$$

Growth of $z(t)$ graphically



$$z(t) \mapsto 0 \quad \frac{\dot{z}(t)}{z(t)} \mapsto \infty$$

$$z(t) \mapsto \infty \quad \frac{\dot{z}(t)}{z(t)} \mapsto \beta.$$

Transition after policy change

Consider an increase in the savings rate or a decrease in the population growth rate. We have seen that those do not change the growth rate of output per worker in steady state. However, they increase its growth rate temporarily during the transition phase. To see this, note

$$y(t) = z(t)^{\frac{\alpha}{1-\alpha}} A(t) \tag{81}$$

$$\frac{\dot{y}(t)}{y(t)} = \frac{\alpha}{1-\alpha} \frac{\dot{z}(t)}{z(t)} + g. \tag{82}$$

During the transition, $\frac{\dot{K}(t)}{K(t)} = \frac{s}{z(t)} - \delta > n + g$ and, hence, the capital-to-output ratio is increasing.

Calculating the growth rate during the transition

$$\frac{\dot{y}(t)}{y(t)} = \frac{\alpha}{1-\alpha}\frac{\dot{z}(t)}{z(t)} + g. \tag{83}$$

Qualitatively, we know already that $\frac{\dot{z}(t)}{z(t)}$ will be highest directly after the policy change and converge to zero as the economy converges to its steady state. However, we can also compute the exact values. Recall:

$$\frac{\dot{z}(t)}{z(t)} = (1-\alpha)\frac{s}{z(t)} - (1-\alpha)(n+g+\delta) \tag{84}$$

with

$$z(t) = \frac{s}{n+g+\delta} + \left[z(0) - \frac{s}{n+g+\delta}\right]\exp(-\beta t) \tag{85}$$

Calculating the growth rate during the transition II

Combining terms

$$\frac{\dot{y}(t)}{y(t)} = g + \alpha \left[\frac{s}{\frac{s}{n+g+\delta} + \left[z(0) - \frac{s}{n+g+\delta} \right] \exp(-\beta t)} - (n+g+\delta) \right] \quad (86)$$

Dividing numerator and denominator by z^* yields

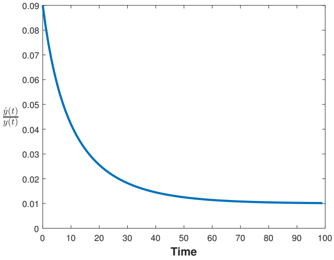
$$\frac{\dot{y}(t)}{y(t)} = g + \alpha \left[\frac{n+g+\delta}{1 + \left[\frac{z(0)}{z^*} - 1 \right] \exp(-\beta t)} - (n+g+\delta) \right] \quad (87)$$

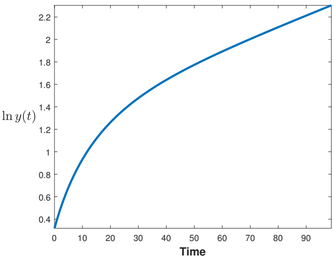
Calculating the growth rate during the transition II

$$\frac{\dot{y}(t)}{y(t)} = g + \alpha \left[\frac{n+g+\delta}{1 + \left[\frac{z(0)}{z^*} - 1 \right] \exp(-\beta t)} - (n+g+\delta) \right] \qquad (88)$$

- When starting in steady state: $z(0) = z^*$, $\frac{\dot{y}(t)}{y(t)} = g$.
- For any $z(0)$, as time passes, $\exp(-\beta t) \mapsto 0$ and $\frac{\dot{y}(t)}{y(t)} \mapsto g$.
- Given t, the smaller is $\frac{z(0)}{z^*}$, the higher is the growth rate.

Graphics of output per worker growth outside steady state





Initially, the economy accumulates capital rapidly, and output per capita is growing rapidly:

$$\frac{\dot{y}(t)}{y(t)} = \frac{\alpha}{1-\alpha}\frac{\dot{z}(t)}{z(t)} + g. \quad (89)$$

As capital accumulation slows down, so does output per capita growth. Note, in the Solow model, a growth rate different from the (exogenous) growth rate g can only result from a change in the capital-to-output ratio.

Introducing human capital

Why human capital

So far, we assume all workers are equally productive across time and countries. However,

- the number of average years of schooling varies substantially within a country over time and across countries.
- the quality of schooling varies substantially within a country over time and across countries.
- within a country at a point in time, income differences across education groups are large suggesting that education matters for productivity.
- you have seen that education is positively correlated to output per capita in the data.

Production

To introduce human capital, we make a small change to the production function:

$$Y(t) = K(t)^{\alpha} \left(A(t)H(t)\right)^{1-\alpha} \tag{90}$$

$$H(t) = \exp(\psi\, u)L(t), \tag{91}$$

where $L(t)$ is the amount of labor, and $H(t)$ is the amount of total human capital. Total human capital not only depends on the amount of labor but also in the time invested in education, u.

Quality of learning

Note that

$$\frac{\partial H(t)}{\partial u} = \psi \exp(\psi\, u) L(t) = \psi H(t) \tag{92}$$

$$\frac{\partial \ln H(t)}{\partial u} = \psi. \tag{93}$$

That is, a change in u translates into ψ percent more human capital. One way to interpret ψ is to think about the quality of the education system for a fixed time spend on it.

Steady state

Again, the capital-to-output ratio has a steady state. To find it, we use again the production function to write:

$$z(t) = \frac{K(t)}{Y(t)} = \frac{K(t)}{K(t)^{\alpha}\left(A(t)H(t)\right)^{1-\alpha}} \tag{94}$$

$$= \left(\frac{K(t)}{A(t)H(t)}\right)^{1-\alpha}. \tag{95}$$

Growth rate of the capital-to-output ratio

As before, we derive again its growth rate:

$$\ln z(t) = (1-\alpha)\ln K(t) - (1-\alpha)(\ln H(t) + \ln A(t)) \tag{96}$$

$$\Rightarrow \frac{\dot{z}(t)}{z(t)} = (1-\alpha)\frac{\dot{K}(t)}{K(t)} - (1-\alpha)\left(\frac{\dot{H}(t)}{H(t)} + \frac{\dot{A}(t)}{A(t)}\right) \tag{97}$$

$$\frac{\dot{z}(t)}{z(t)} = (1-\alpha)\frac{\dot{K}(t)}{K(t)} - (1-\alpha)(n+g) \tag{98}$$

$$\left(\frac{\dot{K}(t)}{K(t)}\right)^{*} = n + g \tag{99}$$

because ψ and u are constant and, hence, $\frac{\dot{H}(t)}{H(t)} = \frac{\dot{L}(t)}{L(t)}$.

Rewriting the capital accumulation equation

Again, we need to find an expression for the growth rate of the capital stock

$$\dot{K}(t) = sK(t)^{\alpha}\left(A(t)H(t)\right)^{1-\alpha} - \delta K(t) \tag{100}$$

$$\frac{\dot{K}(t)}{K(t)} = \frac{s}{z(t)} - \delta \tag{101}$$

Combining the equations and evaluating at steady state

$$n + g = \frac{s}{z^{*}} - \delta. \tag{102}$$

Solving for the steady state

The equation is the same as in the model without education, and, thus, the steady state capital-to-output ratio is the same:

$$z^* = \left(\frac{K(t)}{Y(t)}\right)^* = \frac{s}{n+g+\delta}. \tag{103}$$

Output per capita in steady state

Hence, output per worker in steady state is:

$$Y(t) = K(t)^{\alpha} \left(A(t)H(t)\right)^{1-\alpha} \tag{104}$$

$$\frac{Y(t)}{Y(t)^{\alpha}} = \left(\frac{K(t)}{Y(t)}\right)^{\alpha} \left(A(t)H(t)\right)^{1-\alpha} \tag{105}$$

$$Y(t) = \left(\frac{K(t)}{Y(t)}\right)^{\frac{\alpha}{1-\alpha}} A(t)H(t) \tag{106}$$

$$Y(t)^{*} = \left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} A(t)H(t) \tag{107}$$

$$y(t)^{*} = \left(\frac{Y(t)}{L(t)}\right)^{*} = \left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} A(t)\exp(\psi\, u) \tag{108}$$

Output per capita is increasing in the amount of education. A more educated workforce is more productive and, thereby, allows each worker to produce more.

Capital per capita in steady state

Note that output is increasing one-to-one with $\exp(\psi\, u)$. The reason is that capital (per capita) also increases in $\exp(\psi\, u)$:

$$\left(\frac{K(t)}{Y(t)}\right)^* = \frac{s}{n+g+\delta} \tag{109}$$

and plug in the production function:

$$\left(\frac{K(t)}{K(t)^\alpha \left(A(t)H(t)\right)^{1-\alpha}}\right)^* = \frac{s}{n+g+\delta} \tag{110}$$

$$\left(\frac{K(t)^{1-\alpha}}{L(t)^{1-\alpha}}\right)^* = \frac{s}{n+g+\delta} A(t)^{1-\alpha} \exp(\psi\, u)^{1-\alpha} \tag{111}$$

Hence, in steady state:

$$\left(\frac{K(t)}{L(t)}\right)^* = \left(\frac{s}{n+g+\delta}\right)^{\frac{1}{1-\alpha}} A(t) \exp(\psi\, u). \tag{112}$$

Education makes capital more productive leading to a higher capital to capita ratio.

Growth in steady state

We can ask again about the growth rate in steady state. As education is assumed to be constant, nothing really changes:

$$\frac{\dot{z}(t)}{z(t)} = (1-\alpha)\frac{\dot{K}(t)}{K(t)} - (1-\alpha)(n+g) \tag{113}$$

$$\left(\frac{\dot{K}(t)}{K(t)}\right)^* = n+g \tag{114}$$

$$\left(\frac{\dot{k}(t)}{k(t)}\right)^* = g. \tag{115}$$

That is, capital per capita (and output/consumption per capita) grows at the rate of technological progress.

Transition dynamics

We have already seen that changes in population growth rates and saving rates can have rich transition dynamics for output per worker. We can now also analyze changes in education. In general, output per worker is:

$$Y(t) = \left(\frac{K(t)}{Y(t)}\right)^{\frac{\alpha}{1-\alpha}} A(t)L(t)\exp(\psi u) \tag{116}$$

$$y(t) = \frac{Y(t)}{L(t)} = \left(\frac{K(t)}{Y(t)}\right)^{\frac{\alpha}{1-\alpha}} A(t)\exp(\psi u). \tag{117}$$

As the education variables are constant, the growth rate of output per worker is again

$$\frac{\dot{y}(t)}{y(t)} = \frac{\alpha}{1-\alpha}\frac{\dot{z}(t)}{z(t)} + g. \tag{118}$$

Transition dynamics II

As we have seen before, the dynamic equation for the capital-to-output ratio is the same. Therefore, its solution is also the same:

$$z(t) = \frac{s}{n+g+\delta} + \left[z(0) - \frac{s}{n+g+\delta} \right] \exp(-\beta t), \qquad (119)$$

which is again independent of education variables. Hence, one may be tempted to think that changes in education variables imply no transition dynamics, i.e., the economy jumps directly to its new steady state.

Transition dynamics III

This conclusion is wrong. As we have seen, increasing the time spend in education, u, or the quality of education, ψ, in a period $t = 0$ increases output in period 0:

$$Y(0) = K(0)^{\alpha} \left(A(0)L(0)\exp(\psi\, u)\right)^{1-\alpha}. \tag{120}$$

Hence,in period 0, the capital-to-output ratio falls, and $z(0) < z^* = \frac{s}{n+g+\delta}$. As a result, the capital-to-output ratio will grow over time back to its steady state:

$$z(t) = \frac{s}{n+g+\delta} + \left[z(0) - \frac{s}{n+g+\delta}\right]\exp(-\beta t), \tag{121}$$

implying that output per worker grows quicker than in steady state:

$$\frac{\dot{y}(t)}{y(t)} = \frac{\alpha}{1-\alpha}\frac{\dot{z}(t)}{z(t)} + g. \tag{122}$$

Transition dynamics: intuition

An increase in education increases the marginal product of capital:

$$MPK(t) = \alpha K(t)^{\alpha-1} \left(A(t)H(t)\right)^{1-\alpha} \tag{123}$$

$$= \frac{\alpha}{\frac{K(t)}{Y(t)}}. \tag{124}$$

As before, when the marginal product of capital is higher than in steady state, the net return per unit of investment per capital, $\frac{s}{\alpha}MPK(t) - \delta$, is higher than the growth rate of output in steady state, $(n+g)$, leading to an increase in the capital to output ratio.

Transition dynamics: a critical assessment

Critical for the above analysis is that output responds in period 0:

$$Y(0) = K(0)^{\alpha} \left(A(0)L(0)\exp(\psi\, u)\right)^{1-\alpha}, \tag{125}$$

but capital evolves slowly

$$\dot{K}(t) = sK(t)^{\alpha} \left(A(t)H(t)\right)^{1-\alpha} - \delta K(t) \tag{126}$$

leading to an initial fall in the capital-to-output ratio. However, it is unlikely that human capital will jump to its new level instantaneously with the reform as educating the workforce takes time. Later, we will see a model where human capital also needs to build over time.

How much *should* we save

How much should we save

So far, we take the savings rate as given. One may ask whether there is an optimal savings rate. One possibility to define optimal is the savings rate that maximizes long-run consumption per worker:

$$\left(\frac{C(t)}{L(t)}\right)^* = (1-s)\left(\frac{Y(t)}{L(t)}\right)^* = (1-s)\left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} A(t)\exp(\psi\, u) \tag{127}$$

The resulting savings rate is referred to as the golden rule s_{Gold}.

The golden rule

Taking the first order condition yields

$$s_{Gold} = \alpha. \tag{128}$$

Intuition: The more important is capital in the production function the more we should save.

The economics behind the golden rule

Recall that in steady state, the marginal product of capital is

$$MPK^* = \alpha \frac{n+g+\delta}{s}. \tag{129}$$

A higher savings rate leads to more capital in steady state and, hence, a lower marginal product. At the Golden rule,

$$MPK^* - \delta = n + g. \tag{130}$$

The net marginal gain of an additional unit of capital need to equal its "marginal costs" (the marginal savings required to keep the capital-to-output ratio constant in steady state).

Are we saving enough?

To assess whether Spain has a savings rate consistent with the Golden rule, consider the following data facts

1. The capital output ratio is 2.75: $k = 2.75y$.
2. Capital depreciation is 10 percent of yearly output: $\delta k = 0.1y$.
3. The capital share of income is 30%
4. Output growth is 3%.

Are we saving enough? II

Combining 1 and 2 tells us that the depreciation rate is 3.6%.

According to our model, 3 implies $MPK^* k = 0.3y$. Combining with 1 we have $MPK = 0.11$.

According to our model, 4 implies $n + g = 0.03$.

Hence, $MPK > n + g + \delta$, i.e., we save to little.

Can we rationalize a high MPK

We can draw two possible conclusions from a high MPK:

1. Either we need to change savings incentives. Reforming the pension system is one aspect economists have advocated.
2. Or optimizing long-run consumption per worker is not optimal. If we discount future consumption relative to today's consumption, the Golden rule is not optimal. A yearly discount rate of over 4% would be needed to explain the high capital returns.

The empirics of the Solow model

Results and predictions: steady state

We have seen that the Solow model is consistent with all Kaldor facts. Beyond that, it makes some key predictions about the level of output per worker across countries:

$$\left(\frac{Y(t)}{L(t)}\right)^{*} = \left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} \exp(\psi\, u) A(t) \tag{131}$$

- Increasing in the capital-to-output ratio (endogenous to the model):
  - increasing in the savings rate: $s = \frac{I}{Y}$.
  - decreasing in the population growth rate.
- Increasing in education levels/quality (exogenous to the model).
- Increasing in productivity (exogenous to the model).

A quantitative assessment

You have already discussed the empirical approach of Mankiw et al. (1992). In fact, their reduced-form model follows directly from the Solow model. Assuming that all countries are in steady state, they start from:

$$\left(\frac{Y(t)}{L(t)}\right)^* = A(t)\exp(\psi\, u)\left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} \tag{132}$$

$$\ln y(t) = \ln A(0) + gt + \frac{\alpha}{1-\alpha}\ln s - \frac{\alpha}{1-\alpha}\ln(n+g+\delta) + \psi\, u \tag{133}$$

$$\ln y(t) = \beta_0 + \beta_1 \ln s + \beta_2 \ln(n+0.05) + \beta_3 u + \epsilon(t). \tag{134}$$

Assuming that $\ln A(0) + gt = \beta_0 + \epsilon$, i.e., the level of technology is random across countries, and $g + \delta = 0.05$ they estimate the equation by linear OLS.

A quantitative assessment

- As you have discussed, they find that differences in savings rates, population growth rates, and years of schooling explain almost 80% in cross-country variation in GDP per capita.
- However, you have also seen that development accounting suggests a very different result: almost all cross-country variation in GDP per capita comes from TFP differences. Differences in capital-to-output ratios or differences in the years of schooling explain relatively little.
- Hence, we will need a theory of endogenous TFP.

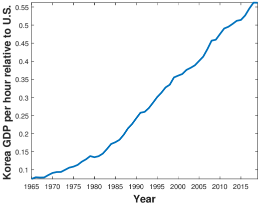
Results and predictions: transition dynamics

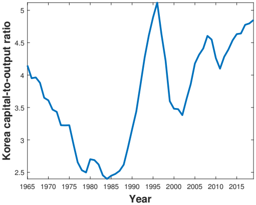
The Solow model provides a compelling theory behind growth miracles:

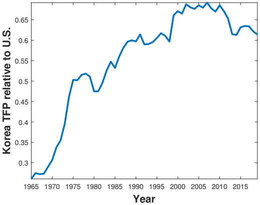
$$\frac{\dot{y}(t)}{y(t)} = \frac{\alpha}{1-\alpha}\frac{\dot{z}(t)}{z(t)} + g. \tag{135}$$

- Reforms increasing the capital-to-output ratio (endogenous to the model):
  - Reforms increasing the savings rate.
  - Reforms decreasing the population growth rate.
- Reforms increasing education levels that result in a sudden decline and subsequent growth in the capital-to-output ratio (endogenous to the model).
- Reforms increasing productivity growth (exogenous to the model).

A quantitative assessment







Again, having a theory of endogenous TFP is crucial to understand the data.

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